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Topological Spin–Statistics Relation in Quantum Field Theory

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The concepts of Bopp–Haag multiple-valued quantization and Dirac–Finkelstein–Rubinstein rubber bands are refined and abstracted in order to develop a topological theory of the connection between spin and statistics in a general framework of quantum field theory.

1. INTRODUCTION

It is generally believed that *spin-odd-half particles* are *fermions* and *spin-integer-particles* are *bosons.* In many textbooks on atomic physics it is stated that the *Pauli principle* for electrons (resp. protons and neutrons)—though postulated in quantum mechanics—can be rigorously proven as a part of the discussion of the connection between spin and statistics in relativistic quantum field theory. Unfortunately, the well-known proof and its generalizations, which are based on analyticity properties of many-point vacuum expectation values of field operator products (see, e.g., Streater and Wightman, 1964), do *not* cover theories including massless gauge particles such as *quantum electrodynamics* (however, see Buchholz and Epstein, 1985). Hence we may conclude that the Pauli principle is not yet well understood. On the other hand spin and statistics may be defined in topological terms—at least in certain models, e.g., configuration space models, kink models, monopole models, etc. (Finkelstein, 1955, 1966, 1969; Finkelstein and Misner, 1959; Finkelstein and Rubinstein, 1968; Finkelstein and Williams, 1984; Polyakov, 1974; 'tHooft, 1974; Goldhaber, 1976; Hasenfratz and 'tHooft, 1976; Jackiw and Rebbi, 1976; Leinaas and Myrheim, 1977). For example, the newly discovered strange spin and

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statistics quantum numbers of two-dimensional solid state physics (fractional quantum Hall effect, vortices in thin films of liquid helium, quasiparticle excitations in two dimensional quantum antiferromagnets related to systems exhibiting high-temperature superconductivity) can be interpreted and related as topological invariants (Halperin, 1984; Schrieffer, 1986; Anderson, 1987; Kalmeyer and Laughlin, 1987; Kivelson *et al.*, 1988; Laughlin, 1988; Polyakov, 1987). Therefore it is an exciting adventure to start up a research program with the aim to prove a more general *topological spin and statistics theorem* covering all branches.

The topological interpretation of spin and statistics quantum numbers has some history and is based on an idea now called *multiple-valued quantization.* It is useful to reconsider this concept, which first was discussed by Bopp and Haag in the setup of spin models, later extended by Finkelstein *et al.* to the framework of kink theories (Bopp and Haag, 1950; Haag, 1952; Finkelstein, 1955, 1966, 1969; Finkelstein and Misner, 1959; Finkelstein and Rubinstein, 1968; Finkelstein and Williams, 1984; Aharonov and Bohm, 1959; Schulman, 1968; Williams, 1970; Williams and Zvengrowski, 1977; Laidlaw and De Witt, 1971; Dowker, 1972; Wu and Yang, 1975; Leinaas and Myrheim, 1977; Friedman and Sorkin, 1980; Tarski, 1980; Sorkin, 1983; Berry, 1984; Wu, 1984).

In this paper we start by investigating quantum mechanics in Schrödinger representation, adopting a philosophy only admitting *position* to be viewed as an observable quantity. We strictly distinguish between the kinematical and the dynamical approach. In the first one we do not need any reference to continuity or differentiability of the wave function and nevertheless have topological obstructions against a description by a globally defined wave function. Some of these survive even in the second one, which is well known and suitably formulated in a fiber-bundle-theoretic language. We demonstrate that those rigid kinematical superselection quantum numbers may also appear in quantum field theory, where continuity and differentiability of wave functions are ill defined. These correspond to occurrence of multiplicative quantum numbers such as spin and statistics. The main goal of our research program is the implementation of the Finkelstein-Rubinstein mechanism in a general quantum field-theoretic context. We propose a replacement of the topological rubber band techniques familiar from kink models by certain algebraic operations and prove that *skyrmions* can be treated this way. We argue that the *a priori* exclusion of the diagonals in quantum mechanical toy models of indistinguishable point particles may be replaced by the noncommutativity of charge-carrying fields with partially coinciding supports at fixed time. We emphasize the meaning of the *existence of conjugated charges* for the connection between spin and statistics (Feynman, 1986).

2. MULTIPLE-VALUEDNESS IN QUANTUM MECHANICS: SPIN, STATISTICS, AND PARTICLE–ANTIPARTICLE MODELS

The formulation of quantum mechanics in Schrödinger representation on a topological nontrivial configuration manifold Q has a long history. The standard approach of today uses the mathematical theory of fiber bundles, but it is not immediately clear what mathematical structures are natural, i.e., related to the measuring process. Evidently, this is due to a lack of a clear distinction between *kinematical* and *dynamical* classifications. It is the aim of the following subsection to clarify this issue.

2.1. The Measurement of Localization in Configuration Space

It is a good standpoint to assume that in quantum mechanics only *position* is a measurable quantity. Though in principle thought to be observable, *momentum* is only measured indirectly, namely in time-of-flight velocity measurements or in interference setups. In reality, the former and the latter are nothing but position measurements.

Once we have accepted the distinguished role of position, we also must accept that the reconstruction of states, wave functions, and potentials is done from the knowledge of the time-dependent probability density and field strengths alone. One immediately encounters the following question: How does one have to describe effects of potentials or path-dependent phase factors such as the Aharonov–Bohm or (adiabatic) Berry phase effects (Aharonov and Bohm, 1959; Berry, 1984)?

2.1.1. Kinematical Obstructions

In a Schrödinger picture and representation a *pure state* at a fixed time is assumed to be represented by a space-dependent wave function $\psi\colon Q \to \mathbf{C}$ defined on the configuration manifold Q. Two wave functions ψ and φ are physically equivalent if and only if they differ by a nonvanishing complex factor:

$$\psi(q) = g\varphi(q) \tag{1}$$

Since localization measurements are performed only in neighborhoods $U \subset Q$ of points $q \in Q$, it is natural to generalize this notion of a pure state: Given an open contractible cover $U = \{U_i\}$ of Q (i.e., a covering by open contractible sets, whose intersections are also contractible) we attach to each U_i a *local wave function* $\psi_i\colon U_i \to \mathbf{C}$ obeying the compatibility conditions

$$\psi_i|_{U_i \cap U_j}(q) = h_{ij}\psi_j|_{U_i \cap U_j}(q) \tag{2}$$

where $q \in U_i \cap U_j$, $U_i \cap U_j$ is nonempty, and $h_{ij} \in \mathbf{C}\backslash\{0\}$, the latter expression denoting the multiplicative Abelian group of nonvanishing complex numbers. The *transition constants* h_{ij} obey a 1-Čech-*cocycle* condition:

$$h_{ij}h_{jk} = h_{ik} \tag{3}$$

It is easy to see that for contractible configuration manifolds this extended "localized" notion of a pure state coincides with the original one, but in case of topological nontrivial manifolds we get new classes of states which are suitably characterized by the nontrivial elements of the 1-Čech-*cohomology group* $H^1(Q, \mathbf{C}\backslash\{0\})$ *with constant coefficients in* $\mathbf{C}\backslash\{0\}$ (see, e.g., Hirzebruch, 1966). In our case this means that the equivalence classes of states are labeled by the characters of the fundamental group:

$$H^1(Q, \mathbf{C}\backslash\{0\}) = \pi_1(Q) \tag{4}$$

What does "equivalence" mean physically? States belonging to different topological classes cannot interfere.

In such a case we cannot find system $\{(U_i, \psi_i)\}$, $\{(U_i, \varphi_i)\}$ of wave functions representing the states ω_ψ, ω_φ for which the linear combinations yield consistent compatibility conditions. Then only mixtures are definable, and as an interesting aside we note that a topology change of the configuration space possibly develops pure states into mixtures and vice versa, i.e., gives rise to a "loss of quantum coherence"[2] (Anderson, 1987).

A local system $\{(U_i, \psi_i)\}$ of wave functions on Q belonging to the cohomology class $\chi \in H^1(Q, \mathbf{C}\backslash\{0\})$ may be represented by a globally defined wave function $\tilde{\psi}_\chi$ defined on the universal covering space $\tilde{Q}$, equivariant with respect to the character χ of the fundamental group $\pi_1(Q)$:

$$\tilde{\psi}_\chi(a\tilde{q}) = \chi(a)\tilde{\psi}_\chi(\tilde{q}) \tag{5}$$

where $a \in \pi_1(Q)$. Given a system $\{(U_i, s_i)\}$ of continuous sections $s_i\colon Q \to \tilde{Q}$, we are able to reconstruct the local system by setting

$$\psi_i(q) = \tilde{\psi}_\chi(s_i(q)) \tag{6}$$

2.1.2. Kinematical Obstructions and Generalized Pure States

As Greenberg and Messiah (1964, 1965) pointed out, there is no reason to define a pure state as a ray or one-dimensional subspace of the Hilbert space. There may exist physical systems whose pure states are described by finite-dimensional, say n-dimensional, subspaces. In Schrödinger representation these may be represented by tuples of n linear independent wave functions. A general linear transformation transforms one n-tuple into

[2]If you cut a part out of a physical system, you will get a loss of information transmuting a wave function into a density matrix. What has been discussed recently is the possibility that this can occur dynamically in quantum gravity.

another physically equivalent one. As a consequence we have to deal with systems of constant transition matrices $\in \mathbf{GL}(n, \mathbf{C})$, when we wish to localize our notion of a pure state as done in the previous case. The classes of states are given by the non-Abelian 1-Čech-cohomology coinciding, according to a well-known theorem of algebraic topology, with the set of equivalence classes of representations of the fundamental group[3] $\pi_1(Q)$,

$$H^1(Q, \mathbf{GL}(n, \mathbf{C})) = \mathrm{Hom}(\pi_1(Q), \mathbf{GL}(n, \mathbf{C}))/\mathbf{GL}(n, \mathbf{C}) \tag{7}$$

In analogy to the one-dimensional case, we are allowed to represent a system of multicomponent wave functions by a globally defined multicomponent wave function obeying an equivariance condition:

$$\tilde{\psi}^{\mu}_{D}(a\tilde{q}) = \sum_{\nu=1}^{n} D^{\mu\nu}(a)\tilde{\psi}^{\nu}_{D}(\tilde{q}) \tag{8}$$

with $[D] \in \Sigma(\pi_1(Q))$ and Σ is the dual object of the group considered (i.e., the set of equivalence classes of all unitary irreducible representations). According to the decomposition of the regular representation of $\pi_1(Q)$ into irreducible components, the Hilbert space

$$H = \mathscr{L}^2(\tilde{Q}, \mathbf{C}) \tag{9}$$

of the "covering wave functions" decomposes into invariant subspaces:

$$H = \bigoplus_{[D] \in \Sigma(\pi_1(Q))} m_D H_D \tag{10}$$

where m_D denotes the multiplicity of the representation D. I summarize: A generalized pure state in a Hilbert space H_D associated to an n-dimensional irreducible representation D of $\pi_1(Q)$ is nothing but an n-dimensional subspace of H_D.

Remark. Here I have assumed implicitly that $\pi_1(Q)$ is compact. In the case of a noncompact Abelian fundamental group, we replace (10) by a direct integral (or a direct sum of direct integrals), whereas the classification of the multidimensional representations of noncompact, non-Abelian fundamental groups (e.g., braid groups) is a highly active research topic of mathematics (cf. below).

2.1.3. Dynamical Obstructions

If one adopts the point of view that in quantum mechanics localization plays a preferred role, then the relevant information about a state will be

[3]Hom(A, B) denotes the set of homomorphisms from A to B.

the probability density ρ in configuration space Q as a function of time $\mathbf{R}$.[4] In this sense two wave functions which differ by a space-time-dependent phase factor may be considered as equivalent.[5]

In the local formulation taking a covering of *configuration space-time* $Q\times\mathbf{R}$ by contractible open sets U_i it will be natural to allow that each h_{ij} appearing in the compatibility condition is a *transition function*, i.e., a smooth function from $Q\times\mathbf{R}$ to the complex numbers of modulus 1 or, if we drop the normalization of the probability density, to the nonvanishing complex numbers $\mathbf{C}\backslash\{0\}$. Physically, the smoothness conditions is a consequence of a selection of "good states"—states in which the momentum is bounded.

The characterization of different classes of wave function systems understood in this sense immediately leads to 1-*Čech-cohomology group* $H^1(Q\times\mathbf{R}, \underline{\mathbf{C}\backslash\{0\}})$ with smooth $\mathbf{C}\backslash\{0\}$-valued functions as coefficients.

The computation of the cohomology group

$$H^1(Q\times\mathbf{R}, \underline{\mathbf{C}\backslash\{0\}}) = H^1(Q, \underline{\mathbf{C}\backslash\{0\}}) = H^2(Q, \mathbf{Z}) \tag{11}$$

leads to a standard result of algebraic topology (see, e.g., Switzer, 1975):

$$H^2(Q, \mathbf{Z}) = \mathbf{Z}^{b^2} \oplus \mathrm{Tors}\, \frac{\pi_1(Q)}{[\pi_1(Q), \pi_1(Q)]} \tag{12}$$

where b^p is the pth Betti number of Q and Tors$(\cdots)$ denotes the torsion subgroup (the subgroup of all finite-order elements) of the Abelization of the fundamental group ($[a, b] = aba^{-1}b^{-1}$).

Relation (11) classifies topologically inequivalent principal $\mathbf{C}\backslash\{0\}$-fiber bundles $E \to^{\pi} Q\times\mathbf{R}$.[6] In analogy to the kinematical discussion, we are allowed to represent a local system of wave function by a globally defined equivariant wave function defined on the bundle:

$$\hat{\psi}(g\hat{q}) = g\hat{\psi}(\hat{q}) \tag{13}$$

where $q \in \mathbf{U}(1)$ and $\hat{q} \in E$. Sorkin (1983) calls such a bundle a *gauge space.*

[4] $\rho(q, t)$ can be thought of as a state *sub specie aeternitatis*, i.e., a quantum mechanical state combined with the associated evolution law. Note that a separation of the kinematical and dynamical parts of information is only possible in some very special cases.

[5] This has much to do with the question of how one gets information about the phase of the wave function by localization measurements at different times, or, more formally, how one gets a reconstruction algorithm yielding both the phase of the time-dependent wave function and the time-dependent potentials from the complete knowledge of the time-dependent probability density $\rho(q, t)$ and time-dependent field strengths alone. It is important to note that only certain $\rho(q, t)$'s are allowed in quantum mechanics, namely those from which *we are able* to reconstruct wave functions and potentials, locally defined in configuration space-time $Q\times\mathbf{R}$ and obeying local Schrödinger equations.

[6] To formulate a dynamical law, it is necessary to introduce a connection in the bundle.

Note that a generalization of the dynamics à la Greenberg-Messiah is possible and leads to the discussion of the topological invariants (Chern classes) of more general principal fiber bundles.

2.1.4. *Kinematical Obstructions versus Dynamical Obstructions*

The group of kinematical classes of states $H^1(Q, \mathrm{U}(1))$ is mapped into the group of dynamical classes $H^2(Q, \mathbf{Z})$ via the long exact cohomology sequence (see, e.g., Switzer, 1975)

$$\cdots \to H^1(Q, \mathbf{R}) \to H^1(Q, \mathrm{U}(1)) \to H^2(Q, \mathbf{Z}) \to H^2(Q, \mathbf{R}) \to \cdots \quad (14)$$

induced by the exact sequence[7]

$$0 \to \mathbf{Z} \hookrightarrow \mathbf{R} \to \mathrm{U}(1) \to 0 \quad (15)$$

Explicitly, the sequence (14) looks like

$$\cdots \to \mathbf{R}^{b^1} \to \widehat{\pi_1(Q)} \to \mathbf{Z}^{b^2} \oplus \operatorname{Tors} \frac{\pi_1(Q)}{[\pi_1(Q), \pi_1(Q)]} \to \mathbf{R}^{b^2} \to \cdots \quad (16)$$

Comparing both groups in question, we are able to determine what kinematical selection rules are also dynamical ones. This leads to the following interesting results:

1. All kinematical superselection rules corresponding to discrete finite-order elements of the character group of the fundamental group survive dynamically, since the image of a finite $\pi_1(Q)^\wedge$ equals $\operatorname{Tors} \pi_1(Q)/[\pi_1(Q), \pi_1(Q)]$. Examples are presented below.
2. It is possible to interpolate dynamically between all kinematical superselection sectors belonging to a continuous family. This may be interpreted as a realization of what Mielnik (1980) calls the *mobility of a physical system.* Typical examples are the idealized Aharonov-Bohm effect and models depending on a θ parameter.
3. In certain systems we have dynamical superselection sectors which cannot be seen kinematically. An example is the Wu-Yang magnetic monopole (Wu and Yang, 1975).

2.2. Bopp-Haag Spin Models and the Corresponding Statistics Models

2.2.1. *Spin Model and Spinors*

Invented nearly 40 years ago, the Bopp-Haag spin model (Bopp and Haag, 1950; Haag, 1952) may be thought of as a model representing a

[7] In an *exact sequence* of homomorphisms the image of the preceding one is the kernel of the succeeding one.

structureless object fixed in space and carrying spin degrees of freedom. It is the prototype of a quantum theory describing a sufficiently asymmetric rigid body and given by

$$Q = \mathbf{SO}(3) \tag{17}$$

Computing the kinematical and dynamical classes, we get

$$H^1(Q, \mathbf{C}\backslash\{0\}) = \mathbf{Z}_2 \tag{18}$$

$$H^2(Q, \mathbf{Z}) = \mathbf{Z}_2 \tag{19}$$

These relations follow from the fact that the fundamental group of $\mathbf{SO}(3)$ is $\mathbf{Z}_2$ and the second Betti numbers of classical Lie groups vanish (Samelson, 1952).

Thus, there are two quantum mechanical classes, kinematically and dynamically stable, which may be represented by χ-equivariant wave functions ψ_χ on the universal covering space $\mathbf{SU}(2)$. The decomposition of the Hilbert space $H = \mathscr{L}^2(\mathbf{SU}(2), \mathbf{C})$ into invariant subspaces

$$H = \underbrace{\bigoplus_{s=0,1,\ldots} \bigoplus_{n=-s}^{+s} H_{s,n}}_{H_{\chi^+}} \oplus \underbrace{\bigoplus_{s=1/2,3/2,\ldots} \bigoplus_{n=-s}^{+s} H_{s,n}}_{H_{\chi^-}} \tag{20}$$

where

$$\dim H_{s,n} = 2s + 1 \tag{21}$$

reproduces the conventional Pauli spinors. The associated orthonormal basis in H is chosen in terms of generalized spherical and hypergeometrical harmonics (Miller, 1968; Wawrzyńczyk, 1984), such that each 3-bein wave function system is decomposable into conventional spinors. Conversely, we are allowed to represent every Pauli spinor as a certain 3-bein wave function system which is uniquely given if the quantum number n is fixed (Bopp and Haag, 1950; Haag, 1952).

One may ask what will happen if we go down to two-dimensional physics. The situation is homotopically equivalent to the Aharonov–Bohm case: Since $\pi_1(\mathbf{SO}(2)) = \mathbf{Z}$ and $b_2(\mathbf{SO}(2)) = 0$, we get a continuous spectrum of kinematical state classes

$$H^1(Q, \mathbf{C}\backslash\{0\}) = \mathbf{U}(1) \tag{22}$$

which may be interpolated between dynamically, since

$$H^2(Q, \mathbf{Z}) = 0 \tag{23}$$

The situation is homotopically equivalent to the Aharonov–Bohm case. Because of

$$\mathbf{Z}_2 = (1, -1\} \subset \mathbf{U}(1) = \{\exp i\varphi\} \tag{24}$$

both class of the three-dimensional case are contained in the two-dimensional case.

What are the Pauli spinors of $\mathbf{SO}(2)$? They are simply complex numbers interpreted as elements of the one-dimensional Hilbert spaces appearing in the decomposition

$$H = \int_{\chi \in \mathbf{U}(1)}^{\oplus} \underbrace{\int_{\mathbf{R}}^{\oplus} H(\mathbf{R})}_{H_\chi} \tag{25}$$

2.2.2. Statistics Models

As Finkelstein and Rubinstein have shown implicitly, it is possible to describe the selection rule between Bose–Einstein and Fermi–Dirac statistics with the help of a configuration space model analogous to the Bopp–Haag spin model. Today these models are put into the framework of *braid theory*, initiated by Artin over 60 years ago (Artin, 1925, 1947, 1959; Fadell, 1962; Fox and Neuwirth, 1962; van Buskirk, 1966; Birman, 1969, 1975; Segal, 1973; McDuff, 1975, 1977; Bloore, 1980).

Let M be a manifold, $\dim M \geq 2$, and let M^N be the associated N-fold product manifold; then the *diagonal set* of M^N is defined to be

$$\Delta_N(M) := \{(m_1, \ldots, m_N) \in M^N \mid \exists_{1 \leq i,j \leq N} m_i = m_j\} \tag{26}$$

The *configuration space of N noncoinciding pointlike particles* is defined as

$$D_N(M) := M^N \backslash \Delta_N(M) \tag{27}$$

The *configuration space of N noncoinciding indistinguishable pointlike particles* is defined as

$$C_N(M) := D_N(M)/\sim \tag{28}$$

where the equivalence of two elements of $D_N(M)$ is fixed through

$$(m_1, \ldots, m_N) \sim (m'_1, \ldots, m'_N) :\Leftrightarrow \{m_1, \ldots, m_N\} = \{m'_1, \ldots, m'_N\} \tag{29}$$

with

$$m_1, \ldots, m_N, m'_1, \ldots, m'_N \in M \tag{30}$$

Denoting by card s the cardinal number of the finite subset $s \subset M$ (discrete subset of points), we may write

$$C_N(M) := \{s \subset M \mid \text{card } s = N\} \tag{31}$$

The fundamental groups of $D_N(M)$ and $C_N(M)$ are called the *pure* resp. *full braid group* of *M*. Both groups are related by

$$\pi_1(C_N(M))/\pi_1(D_N(M)) = \Sigma_N \tag{32}$$

where Σ_N denotes the N-dimensional symmetric group. In case of dim $M > 2$ we have

$$\pi_1(D_N(M)) = \pi_1(M)^N \tag{33}$$

Physics in a three-dimensional *space of perception* is modeled by setting $M = \mathbf{R}^3$. Setting $Q = D_N(\mathbf{R}^3)$, we obtain a quantum mechanical model of the classical Maxwell–Boltzmann statistics (statistics of labeled particles). This toy model is of interest, since because of $\pi_1(D_N(\mathbf{R}^3)) = 0$ it admits only one kinematical state class, but different dynamical classes. Especially for $N = 2$ we get a mechanistic caricature of a *dyon*: Since $D_2(\mathbf{R}^3)$ and $\mathbf{S}^2$ are homotopically equivalent, we have $H^2(D_2(\mathbf{R}^3)) = \mathbf{Z}$.

Yet the fundamental assumption in the quantum statistical description of elementary particles is that they are indistinguishable. Therefore we do better to work on the configuration space

$$Q = C_N(\mathbf{R}^3) \tag{34}$$

The fundamental group of $C_N(\mathbf{R}^3)$ is given by

$$\pi_1(C_N(\mathbf{R}^3)) = \Sigma_N \tag{35}$$

Hence $D_N(\mathbf{R}^3)$ is the universal covering space of $C_N(\mathbf{R}^3)$. Evidently, the elements of $C_N(\mathbf{R}^3)$ are nothing but sets $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, whereas the elements of $D_N(\mathbf{R}^3)$ are ordered sets or N-tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$. On $C_N(\mathbf{R}^3)$ we only encounter two kinematical state classes in correspondence to the fact that Σ_N, $N \geq 2$, has only two characters:

$$\widehat{\Sigma_N} = \mathbf{Z}_2 \tag{36}$$

There are no additional dynamical classes, since the second Betti number of $C_N(\mathbf{R}^3)$ vanishes (Bloore, 1980).

The decomposition of the Hilbert space $\mathscr{L}^2(D_n(\mathbf{R}^3))$ of covering wave functions[8] includes all the invariant subspaces associated to the irreducible multidimensional representations of the symmetric group Σ_N. Therefore, if we introduce a generalized notion of a pure state, we have a description of parastatistics based on braid theory. It is also possible to consider families of configuration spaces localized in certain regions in the space of perception $\mathbf{R}^3$ enabling us to reformulate the famous cluster law, such that we are able to reproduce the discussion of Hartle and Stolt, concluding that paraparticles

[8]Remember that for $Q = C_N(\mathbf{R}^3)$ we have $\tilde{Q} = D_N(\mathbf{R}^3)$.

are ordinary particles just carrying an auxiliary quantum number (Hartle and Taylor, 1969; Stolt and Taylor, 1970*a*,*b*).[9]

An important question is the inclusion or exclusion of diagonal elements. To interpret the statistics signum of a many-particle wave function as a homotopical invariant would be wrong if we included the diagonal elements. It is a very important point that a configuration space of indistinguishable particles on an $(n>2)$-dimensional manifold including the diagonal set does not admit the structure of a topological (resp. differentiable) manifold, so that there are at least technical reasons to exclude the diagonal set. The physical justification for the exclusion comes from the fact that the configuration spaces $C_N(M)$ are embedded naturally into certain field-theoretic configuration spaces (namely those of the kink theories; cf. below), where the homotopical interpretation of statistics will be preserved and diagonals nevertheless are included.

What will happen if we reduce the dimensionality of the space of perception to two? The case of indistinguishable particles has been discussed in the framework of Feynman path integrals by Wu (1984). In order to keep this article readable, I briefly review the discussion.

It is true that the structure of the pure and full braid group of the Euclidean plane $\mathbf{R}^2$ is rather complicated—they are discrete, infinite non-Abelian groups—their characters, however, are easily computed from their presentation and defining relations. This group $\pi_1(C_N(\mathbf{R}^2))$ has N generators $\sigma_1, \ldots, \sigma_{N-1}$, where σ_i may be interpreted as an oriented exchange of the ith and $(i+1)$th particle. The defining relations are given by

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad 1\leq i, j\leq N-2 \tag{37}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad 1\leq i, j\leq N-1, \quad |i-j|\geq 2 \tag{38}$$

Assuming that $\chi(\cdot)$ is a one-dimensional representation, we have from the first relation

$$\chi(\sigma_1) = \cdots = \chi(\sigma_{N-1}) \tag{39}$$

whereas the second relation yields an identity. The corresponding character formula reads

$$\chi(\sigma_i) = e^{-i\theta} \tag{40}$$

with $\theta\in[0, 2\pi]$, such that

$$\pi_1(C_N(\mathbf{R}^2))^{\wedge} = \mathbf{U}(1) \tag{41}$$

[9]In this context it would be interesting to consider the generalized dynamics and the associated important question of how quantum mechanical phase factors (resp. their non-Abelian generalizations) are related to quantum field-theoretic gauge groups (see also Drühl *et al.*, 1970).

To conclude, the simple exchange of two indistinguishable particles alters the wave function by a phase factor fixed through the given kinematical state class. This phenomenon is called *exotic, anomalous, interpolating,* or *fractional statics,*[10] and its is generally accepted that such a type of statistics is experimentally realized for quasiparticles in solid state physics, e.g., fractional quantized Hall effect (Störmer *et al.*, 1983; see also von Klitzing *et al.*, 1980) and has some meaning in the theoretical study of (2+1)-dimensional field theories related to high-T_c superconductors (Anderson, 1987; Polyakov, 1987).[11]

Contrary to this, I am now going to propose a statistics model that is distinguished by a Fermi–Bose alternative even in two space dimensions.

2.2.3. *Particle–Antiparticle Models*

The configuration space of noncoinciding indistinguishable positive and noncoinciding indistinguishable negative pointlike particles of total charge N is defined by[12]

$$C_N^{\pm}(M) := \{(s, t) \subset M \times M \mid \text{card } s - \text{card } t = N\}/\sim \tag{42}$$

where the equivalence of two elements of $\{\cdots\}$ is given by

$$(s, t) \sim (s', t') :\Leftrightarrow s \backslash t = s' \backslash t' \text{ and } t \backslash s = t' \backslash s' \tag{43}$$

Hence the configuration space $C_N^{\pm}(M)$ is topologized in such a way that particles of the same charge sign never collide, while pairs of particles carrying opposite charges may be created or annihilated. In case of an open M the homotopy type of $C_N^{\pm}(M)$ is independent of N.

We are interested in cases where

$$M = \mathbf{R}^n, \qquad n \geq 2 \tag{44}$$

An element of $C_N^{\pm}(\mathbf{R}^n)$ is written as

$$q = (\{\mathbf{x}_1, \ldots, \mathbf{x}_{\mathscr{P}}\}, \{\mathbf{y}_1, \ldots, \mathbf{y}_{\mathscr{N}}\}) \tag{45}$$

[10] By using the generalized notion of a pure state à la Greenberg and Messiah we arrive at the concept of *exotic parastatistics*. Its analysis requires a detailed study of the multidimensional representations of the braid group of the Euclidean plane, for which we do not have a complete theory yet. The associated problems are intimately related to classification problems of type II von Neumann algebra factors and to the study of the algebras introduced by Jones (1985, 1987).

[11] In two space dimensions we also have a quantum analogue to the Maxwell–Boltzmann statistics defined by the configuration space $\pi_1(M_N(\mathbf{R}^2))$. For $N = 2$ we have a situation homotopic equivalent to Aharonov–Bohm.

[12] These spaces were investigated for the first time by McDuff (1975) in a purely mathematical framework without any reference to physical applications.

with

$$N = \mathcal{N} - \mathcal{P} \tag{46}$$

and

$$\mathbf{x}_i \neq \mathbf{y}_j, \qquad \forall i = 1, \ldots, \mathcal{P}; j = 1, \ldots, \mathcal{N} \tag{47}$$

Let us compute the fundamental group of $C_N^{\pm}(\mathbf{R}^2)$: Consider a loop with base point q parameterized by $t \in [0, 1]$. This loop describes a process in which $\mathcal{P}$ particles and $\mathcal{N}$ antiparticles are exchanged and an equal number (say $\mathcal{Q}$) of pair creations and annihilations take place. This loop may be deformed in such a way that all pair creations take place at

$$t = 0 \tag{48}$$

and all pair annihilations at

$$t = 1 \tag{49}$$

at the points $\mathbf{z}_1, \ldots, \mathbf{z}_{\mathcal{Q}}$, which are not allowed to coincide with $\mathbf{x}_1, \ldots, \mathbf{x}_{\mathcal{P}}$ and $\mathbf{y}_1, \ldots, \mathbf{y}_{\mathcal{N}}$. Hence we have a loop describing a process in which $\mathcal{P} + \mathcal{Q}$ particles, localized at $\mathbf{x}_1, \ldots, \mathbf{x}_{\mathcal{P}}, \mathbf{z}_1, \ldots, \mathbf{z}_{\mathcal{Q}}$ and $\mathcal{N} + \mathcal{Q}$ antiparticles, localized at $\mathbf{y}_1, \ldots, \mathbf{y}_{\mathcal{N}}, \mathbf{z}_1, \ldots, \mathbf{z}_{\mathcal{Q}}$, are exchanged. By using standard arguments of braid theory we are able to decompose this loop into "subloops," each one describing a two-particle exchange or a two-antiparticle exchange. Now with help of pair creation and annihilation *each* two-particle exchange (resp. two-antiparticle exchange) is deformable into an *exchange of partners*, a process in which the two particles (resp. antiparticles) are left unexchanged, whereas at another location two particle–antiparticle pairs are created, whose particle members (resp. antiparticle members) are exchanged, and finally the newly configurated pairs are annihilated (see Figure 1).

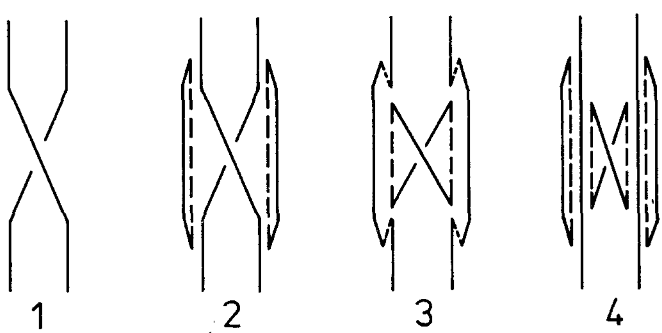

Fig. 1. Deformation of a simple exchange into an exchange of partners.

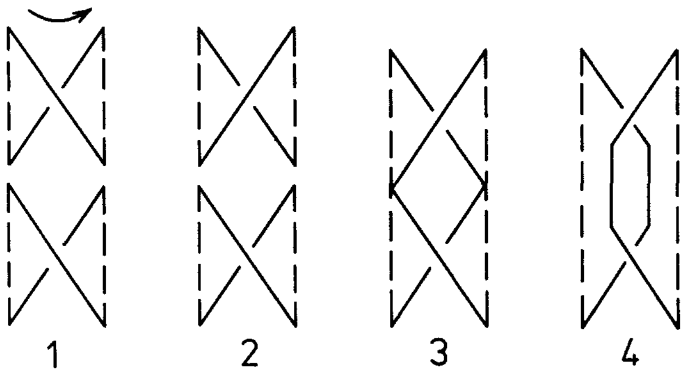

Fig. 2. Triviality of the juxtaposition of two exchanges of partners.

Hence, all nontrivial two-particle exchanges (resp. 2-antiparticle exchanges) are homotopic to each other. Since the juxtaposition of two such exchanges may be deformed into a trivial loop, they define an element of order two in $\pi_1(C_N^{\pm}(\mathbf{R}^n))$ (see Figure 2).

Moreover, every nontrivial two-antiparticle exchange is equivalent to a two-particle exchange (Figure 3), such that we may conclude as follows.

Theorem 2.1. On the configuration space of noncoinciding, indistinguishable positive and noncoinciding, indistinguishable negative pointlike particles of total charge N moving in space of perception $\mathbf{R}^n$, $n \geq 2$, there are exactly two state classes corresponding to the fact that

$$\pi_1(C_N^{\pm}(\mathbf{R}^n))^{\wedge} = \widehat{\mathbf{Z}_2} = \mathbf{Z}_2 \tag{50}$$

In case of two space dimensions the analytic verification is immediately given by the figures and is mechanical, but lengthy. The relation remains true in more than two dimensions, because the exchange of two identical particles always defines a loop of order two.

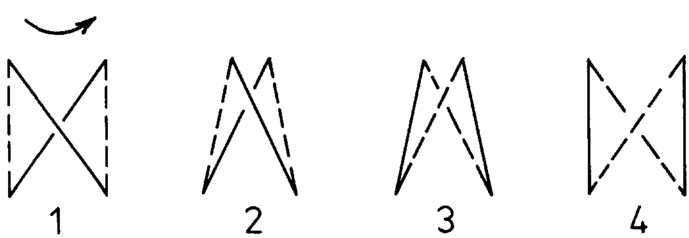

Fig. 3. Homotopy of particle exchange and antiparticle exchange.

To conclude: The introduction of particle–antiparticle pairs reduces the fundamental group of the configuration space in such a way that it is becoming Abelian. Now *parastatistics* is impossible. Evidently this is due to the existence of an integer-valued charge quantum number.

Note that the points of the universal covering space $D_N^{\pm}(\mathbf{R}^n)$ of $C_N^{\pm}(\mathbf{R}^n)$ may be written as

$$\tilde{q} = [((\mathbf{x}_1, \ldots, \mathbf{x}_{\mathscr{P}}), (\mathbf{y}_1, \ldots, \mathbf{y}_{\mathscr{N}}))], \qquad N = \mathscr{P} - \mathscr{N} \tag{51}$$

The square brackets label a class of $(\mathscr{P}, \mathscr{N})$-tuples with respect to the equivalence relation generated by

$$((\mathbf{x}_1, \ldots, \mathbf{x}_{\mathscr{P}}), (\mathbf{y}_1, \ldots, \mathbf{y}_{\mathscr{N}})) \sim ((\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(\mathscr{P})}), (\mathbf{y}_{\tau(1)}, \ldots, \mathbf{y}_{\tau(\mathscr{N})})) \tag{52}$$

if and only if

$$(\text{sign } \sigma)(\text{sign } \tau) = 1 \tag{53}$$

and

$$\begin{aligned} &((\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_{\mathscr{P}}), (\mathbf{y}_1, \ldots, \mathbf{y}_k, \ldots, \mathbf{y}_{\mathscr{N}})) \\ &\quad \sim ((\mathbf{x}_1, \ldots, \cancel{\mathbf{x}_k}, \ldots, \mathbf{x}_{\mathscr{P}}), (\mathbf{y}_1, \ldots, \cancel{\mathbf{y}_k}, \ldots, \mathbf{y}_{\mathscr{N}})) \end{aligned} \tag{54}$$

if and only if

$$\mathbf{x}_k = \mathbf{y}_k \tag{55}$$

2.2.4. Spin-Statistics Models

Since both the alternative between integer and half-odd spin and the alternative between Bose–Einstein and Fermi–Dirac statistics are described by simple mechanistic toy models, one could ask the question whether this may be accomplished in the case of the *connection between spin and statistics.* For this purpose we have to consider configuration spaces which incorporate eigenrotation degrees of freedom as translation degrees of freedom.

A very simple mechanistic model for spin and statistics is defined by the movement of N noncoinciding indistinguishable pointlike particles in space of perception $\mathbf{R}^3$ to which there is attached one 3-bein $\in \mathbf{SO}(3)$ each. In this case "noncoinciding" means that two 3-beins are not allowed to be attached to the same point.

The associated configuration space is given by

$$C_{N\mathbf{SO}(3)}(\mathbf{R}^3) := \mathscr{M}_{N\mathbf{SO}(3)}(\mathbf{R}^3)/\sim \tag{56}$$

$$:= \{(\mathbf{SO}(3) \times \mathbf{R}^3)^N \backslash \Delta_{N\mathbf{SO}(3)}(\mathbf{R}^3)\} \backslash \sim \tag{57}$$

where the "modified diagonal set" is defined to be

$$\Delta_{N\mathbf{SO}(3)}(\mathbf{R}^3) := \{((\mathbf{e}_1, \mathbf{x}_1), \ldots, (\mathbf{e}_N, \mathbf{x}_N)) \in (\mathbf{SO}(3) \times \mathbf{R}^3)^N \mid \exists_{1 \leq i,j \leq N} \mathbf{x}_i = \mathbf{x}_j\} \tag{58}$$

and the equivalence relation is given by

$$((\mathbf{e}_1, \mathbf{x}_1), \ldots, (\mathbf{e}_N, \mathbf{x}_N)) \sim ((\mathbf{e}'_1, \mathbf{x}'_1), \ldots, (\mathbf{e}'_N, \mathbf{x}'_N)) \tag{59}$$

$$:\Leftrightarrow \{(\mathbf{e}_1, \mathbf{x}_1), \ldots, (\mathbf{e}_N, \mathbf{x}_N)\} = \{(\mathbf{e}'_1, \mathbf{x}'_1), \ldots, (\mathbf{e}'_N, \mathbf{x}'_N)\} \tag{60}$$

with

$$\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{x}'_1, \ldots, \mathbf{x}'_N \in \mathbf{R}^3 \tag{61}$$

$$\mathbf{e}_1, \ldots, \mathbf{e}_N, \mathbf{e}'_1, \ldots, \mathbf{e}'_N \in \mathbf{SO}(3) \tag{62}$$

A continuous exchange of the N 3-beins, which are assumed to be fixed in orientation, is described by means of a path parametrized by $t \in [0, 1]$, which looks as follows:

$$q(t) = \{(\mathbf{e}_1, \mathbf{x}_1(t)), \ldots (\mathbf{e}_N, \mathbf{x}_N(t))\} \tag{63}$$

The initial and final points are given by

$$q(0) = \{(\mathbf{e}_1, \mathbf{x}_1(0)), \ldots, (\mathbf{e}_N, \mathbf{x}_N(0))\} \tag{64}$$

$$q(1) = \{(\mathbf{e}_1, \mathbf{x}_1(1)), \ldots, (\mathbf{e}_N, \mathbf{x}_N(1))\} \tag{65}$$

$$= \{(\mathbf{e}_1, \mathbf{x}_{\sigma(1)}(0)), \ldots, (\mathbf{e}_N, \mathbf{x}_{\sigma(N)}(0))\} \tag{66}$$

By choosing a basepoint fulfilling the condition

$$\mathbf{e}_1 = \cdots = \mathbf{e}_N \tag{67}$$

this path is a loop representing a $\sigma \in \pi_1(C_{N\mathbf{SO}(3)}(\mathbf{R}^3))$. This σ is written as a product of generators obeying the relations of the generators of the symmetric group Σ_N. In contrast to this, a 2π rotation of the 3-bein $\mathbf{e}_i$ at $\mathbf{x}_i$ defines an element of order two $\in \pi_1(C_{N\mathbf{SO}(3)}(\mathbf{R}^3))$, which may be written as $-\mathbf{1}_i$. A complete representation of the group will be obtained if the relations

$$(-\mathbf{1}_i)(\sigma) = (\sigma)(-\mathbf{1}_{\sigma(i)}) \tag{68}$$

$$(-\mathbf{1}_i)^2 = \mathbf{1} \tag{69}$$

are added. The resulting finite group has a very complicated structure and is by no means identical to $\mathbf{Z}_2^N \times \Sigma_N$. For example, in the case $N = 2$ we get the *dihedral group* $\mathbf{D}_4$ of order 8, also called *octic group* (Thomas and Wood, 1980):

$$\pi_1(C_{2\mathbf{SO}(3)}(\mathbf{R}^3)) = \mathbf{D}_4 \tag{70}$$

This is the symmetry group of a square in ordinary 3-space. Its Abelization may be identified with the Kleinian group of order 4. Hence the character group is given by

$$\widehat{\mathbf{D}_4} = \widehat{\mathbf{V}_4} = \mathbf{Z}_2 \times \mathbf{Z}_2 =: \mathbf{Z}_2^{\text{spin}} \times \mathbf{Z}_2^{\text{statistics}} \tag{71}$$

These arguments are easily generalized for arbitrary N. From the relation (68) we get the conditions

$$\chi(-\mathbf{1}_i) = \chi(-\mathbf{1}_j) \tag{72}$$

Thus, we have

$$\pi_1(C_{N\mathbf{SO}(3)}(\mathbf{R}^3))^\wedge = \mathbf{Z}_2^{\text{spin}} \times \mathbf{Z}_2^{\text{statistics}} \tag{73}$$

It is an interesting point that a 2π rotation of *each* 3-bein induces the same phase rotation. The usual relation between spin and statistics may be expressed in such a way that nature admits only those quantizations which are trivial or totally double-valued. Of course, this *ad hoc* restriction does not explain the observed connection between spin and statistics. It would be superfluous if we replaced Q by another configuration space Q' in which a 2π rotation of one particle and a simple exchange of two particles are represented by homotopic loops.[13]

A reduction of the fundamental group is given by the introduction of antiparticles: McDuff made the proposal to generalize the concept of the configuration space of charged particles by assuming that each particle has an inner structure described by some parameter taking values in a parameter manifold (McDuff, 1975). Two oppositely charged particles can be annihilated only if their parameters take equal values. If the parameter manifold is identified with $\mathbf{SO}(3)$, then we see immediately that

$$\pi_1(C^{\pm}_{N\mathbf{SO}(3)}(\mathbf{R}^3)) = \mathbf{Z}_2 \times \mathbf{Z}_2 \tag{74}$$

The fundamental group is Abelized again by the introduction of antiparticles. It is true that the character group remains the same, but now generalized quantizations (i.e., those which are characterized by multidimensional subspaces) are impossible.[14] Nevertheless, even in the model of positive and

[13]In two space dimensions anomalous spin *and* statistics occur and match together in a rather complicated way. Moreover, introducing the generalized notion of a state à la Greenberg and Messiah, we get a highly complex interweaving of anomalous spin, statistics, and parastatistics.

[14]E.g., in the case without conjugated charges a generalized quantization leads to a new concept which may be called "para-spin-statistics."

negative 3-beins spin and statistics are fully independent. In kink field-theoretic configuration spaces this is no longer the case. Why this is so will be discussed in Section 3.

2.2.5. Statistics of Ball-Like Objects versus Statistics of Conelike Objects

In concurrence to the investigation of configuration spaces of pointlike particles, we may consider configuration spaces of small, rigid balls. The homotopic properties of these spaces are insensitive against this modification, such that the results derived above are preserved.

Configuration spaces of ball-like objects may be viewed as a caricature of certain field-theoretic quasi-configuration spaces, namely those that are generated by charge-carrying localized fields. It is not a good approach to interpret the "particles" of the configuration models as the real particles of quantum field theory. Therefore, it is better to speak of *objects.* In this sense kinks, strings, or p-branes are also objects. Remember that particles are nothing but eigenstates of the mass operator $\mathcal{M}^2$.

In general, charge-carrying fields are not localized strictly in the sense that the field operator at a fixed time has its support in a ball-like region. This is only true for charges of the first kind, i.e., charges associated with a global gauge symmetry. In case of charges associated with local gauge symmetries we have to deal with conelike supports and therefore it is useful to inspect the statistics of conelike objects (Buchholz and Fredenhagen, 1982).

It is easy to see that the statistical analysis of conelike objects in space of perception $\mathbf{R}^n$ can be "retracted" to the statistics of ball-like or pointlike objects moving on the boundary sphere $\mathbf{S}^{n-1}$. In three space dimensions the associated toy models are given by $C_N(\mathbf{S}^2)$ [resp. $C_N^{\pm}(\mathbf{S}^2)$]. The representation of $\pi_1(C_N(\mathbf{S}^2))$ is given by the representation of $\pi_1(C_N(\mathbf{R}^2))$ adding the relation

$$\sigma_1\sigma_2\cdots\sigma_{N-1}\sigma_{N-1}\cdots\sigma_2\sigma_1=\mathbf{1} \tag{75}$$

This relation yields an auxiliary condition for the characters, namely

$$\chi(\sigma_1)\chi(\sigma_2)\cdots\chi(\sigma_{N-1})\chi(\sigma_{N-1})\cdots\chi(\sigma_2)\chi(\sigma_1)=\chi(\sigma_i)^{2(N-1)}=1 \tag{76}$$

Therefore we have

$$\chi(\sigma_i)=e^{i\pi m/(N-1)}, \qquad m=0,\ldots,2(N-1) \tag{77}$$

i.e.,

$$\pi_1(C_N(\mathbf{S}^2))\hat{}=\mathbf{Z}_{2(N-1)} \tag{78}$$

The case defined by $C_N^{\pm}(\mathbf{S}^2)$ describes the situation of Abelian gauge charges [with gauge group $\mathbf{U}(1)$ dual to $\mathbf{Z}$].[15] Applying partner exchange techniques, we immediately see that

$$\pi_1(C_N^{\pm}(\mathbf{S}^2))^{\wedge} = \widehat{\mathbf{Z}_2} = \mathbf{Z}_2 \tag{79}$$

i.e., exotic statistics for "electrical charges" in three dimensions is forbidden (!). In contrast to this, exotic statistics for charges of the second kind in two space dimensions is possible in analogy to the situation discussed by Streater and Wilde (1970), who observed anomalous statistics for charges of the *first* kind in *one* space dimension.

Summarizing, one can state that introducing conjugated charges has the same effect as incrementing the number of space dimensions, while swinging over from ball-like to conelike objects is like decrementing the number of space dimensions. This implies that in two space dimensions anomalous statistics may be realized in two different ways, namely either with the aid of ball-like objects *excluding* conjugated charges or with the aid of conelike objects *including* conjugated charges.

3. MULTIPLE-VALUEDNESS IN QUANTUM FIELD THEORY: DIRAC'S STRING GAME, FINKELSTEIN–RUBINSTEIN MECHANISM, AND BEYOND

The main question of this research program is the following: How are the concepts introduced above generalized to fit well into the framework of quantum field theory? Formally this is done in Finkelstein's kink theories by using configuration spaces consisting of classical continuous field configurations topologized by means of the compact open topology (Finkelstein and Misner, 1959; Finkelstein and Rubinstein, 1968; Finkelstein, 1966) In kink theories the occurrence of spin and statistics quantum numbers is due to the multiple connectivity of these field-theoretic configuration spaces. Evidently, this is a rather special approach, since it covers only models in which the charge quantum numbers appear as homotopic invariants. In quantum field theory one often deals with Noether charges, i.e., charges associated with a gauge symmetry. Duality transformations, disorder-order puzzling, etc., indicate that there is no fundamental difference between both types of charges. Therefore one is asked to formulate the main principles in a framework in which both homotopic and Noether charges are treated on the same footing. Such a framework exists: It is the

[15]The investigation of the case generalized à la Messiah and Greenberg requires a detailed study of the multidimensional representations of the braid groups of the 2-sphere and would be very interesting, since it is related to the confinement problem.

algebraic quantum field theory (AQFT) proposed by Haag and Kastler; a introductory review is given in Haag (1970). The discussion of superselection rules initiated by Borchers (1965), improved by Doplicher *et al.* (1969*a*,*b*, 1971, 1974; Doplicher and Roberts, 1972; see also Roberts, 1975), later refined by Buchholz and Fredenhagen, 1982),[16] clarifies the foundations of how charge and statistics quantum numbers come in using only the first principles of quantum theory, relativity, and locality.

3.1. Replacing the Configuration Space: Quasi-Configuration Spaces, the Kinematical Phase, and Berry's Phase

In algebraic quantum field theory one never refers to a classical configuration space. AQFT is a true relativistic quantum theory, mathematically well defined. The main disadvantage, however, is a complete lack of a dynamical description of elementary particle interactions. At the moment, it only provides a framework for the discussion of superselection rules, thermodynamic properties, and the general covariance structure of quantum field theory.

We are interested in the global topological structure of field theory in a certain sense. The Dirac string model to be explained later indicates that the connection between spin and statistics can be seen through a very simple topological mechanism unfortunately not realized in the familiar analytic proofs. However, the string mechanism is naturally contained in the discussion of kink models. The generalization of the Dirac, resp. Finkelstein-Rubinstein, mechanism inevitably leads to the AQFT framework, because it probably contains the topological structure we need.

The first step toward a topological spin-statistics theorem is the reformulation of multiple-valued quantization without using a classical configuration space. How can this be achieved?

Quantum mechanics is usually formulated in an infinite-dimensional Hilbert space H. The manifold of pure states $|\Psi\rangle\langle\Psi|$ is identified with the 1-Grassmann manifold $\mathbf{G}_1(H)$, i.e., the manifold of rays in H, whereas the manifold of state vectors $|\Psi\rangle$ is the 1-Stiefel manifold $\mathbf{V}_1(H)$, i.e., the manifold of complex 1-beins in H (see, e.g., Simms, 1968; Varadarajan, 1968, 1970).

The 1-Grassmann manifold or pure states is the quantum analogue of what is called a configuration space in classical mechanics. If we wish to associate to every state a state vector, we have to specify a section in the $\mathrm{U}(1)$ bundle $\mathbf{V}_1(H) \to^{\pi} \mathbf{G}_1(H)$. This section need not be continuous—the phase choice is completely arbitrary. Replacing the global section by a

[16]Recently a new paper by Fredenhagen *et al.* (1988) appeared dealing with braidlike statistics in the DHR framework. The relation to the configuration models is not entirely clear yet.

system of locally defined sections obeying constant compatibility conditions, we obtain no additional structure.

Why are different quantum systems different? Why are they different although their Hilbert spaces and associated manifolds are isomorphic? To put it bluntly, they are different because kinematics and dynamics *select* states giving them a distinguished role. It is like the effect of constraining forces in analytical mechanics: they are introduced to *define* the physical manifold.

How are we able to construct a submanifold of $\mathbf{G}_1(H)$, called a *quasi-configuration space*, resembling the homotopical structure of Q? Formally, we take the continuous basis of configuration eigenvectors $\{|q\rangle\}$, divide out the phase, and construct a manifold $Q_{\mathrm{uasi}} = \{|q\rangle\langle q|\}$ formally homeomorphic to Q. If we do not want to work in the rigged Hilbert space, we have to smear out the eigenvectors and eigenstates, respectively, such that Q_{uasi} is naturally imbedded in $\mathbf{G}_1(H)$. It is easily seen that systems of locally defined sections obeying constant compatibility conditions lead to superselection sectors of superponable state vectors.

The discussion may be extended including the consideration of dynamics. Let us assume that the localized eigenstates are confined in a box and transported adiabatically. The adiabatic dynamics is governed by a slowly varying Hamiltonian, which controls the transport of the localized eigenstates along a specified path on the manifold. If we introduce local systems of smooth sections fulfilling smooth compatibility conditions and the associated local potentials, we get a recapitulation of the dynamical discussion on the configuration space.

The formalism exposed so far may be interpreted as a certain generalization of Berry's (1984) phase formulated on a quasi-configuration space homeomorphic to a classical configuration space. In most cases the quasi-configuration spaces under investigation are defined by the parameter-dependent Hamiltonian appearing in the energy eigenvalue equation. To discuss discrete finite-order superselection rules it suffices to work only with *kinematical phases.* They do not need any reference to smoothness properties and therefore are suited to be used in quantum field-theoretic considerations.

3.2. Quantum Field Theory and Quasi-Configuration Spaces: The Doplicher–Haag–Roberts Statistics and Spin Parameters as Topological Quantum Numbers

In this subsection I give a short sketch of how the concept of multiple-valued quantization may be generalized to the framework of algebraic quantum field theory. For a self-contained introduction to AQFT see the easy-to-read lectures of Haag (1970).

A concrete motivation is given by the following observation: In the discussion of the structure of superselection rules by Doplicher, Haag, and Roberts the statistics of a charge sector (of the first kind) is characterized by a so-called ε parameter, which essentially is interpreted as a signum of a permutation; for details see Doplicher *et al.*, 1969*a*,*b*, 1971, 1974; Doplicher and Roberts, 1972; Roberts, 1975). Doplicher *et al.* show that in space dimensions greater than or equal to two this parameter can take only the values +1 (Bose) or −1 (Fermi). It is true that in two space dimensions this fact is astonishing when we think of the possibility of fractional statistics à la Wu (1984), but it is by no means surprising when we compare this formalism with the particle-antiparticle configuration space models.

What structures of these configuration space models can be recovered in algebraic quantum field theory? Is it true that the ε parameter labels something like a kinematical state class? I think it is. Hence, we have a technique which may have much use in upcoming research.

The main goal of our research program is a definition of suitable quantum field-theoretic configuration spaces which enable us to prove a *topological spin-statistics theorem.*

The first step toward such a theorem is as follows.

We restrict our attention to states of spacelike disjoint localized charges of the first kind, which are generated by localized automorphisms of a C^*-algebra $\mathscr{A}$ of quasilocal observables in Minkowski space (Haag, 1970). Let us especially consider the case characterized by a group $\mathbf{Z}$ of charge sectors [dual to an Abelian global gauge group $\mathbf{U}(1)$].

Let us summarize two main points of the discussion of Doplicher *et al.* (1969):

1. An important consequence of *local commutatively* is the fact that spacelike disjoint localized automorphisms are commuting:

$$\gamma_1\gamma_2 = \gamma_2\gamma_1 \tag{80}$$

2. The *DHR statistics parameter* is defined by the observation that spacelike disjoint localized automorphisms leading from the vacuum to the same sector are transformed in to each other by an inner morphism of the C^*-algebra of observables:

$$\gamma_2 = \sigma_{U_{21}}\gamma_1 \tag{81}$$

with

$$\sigma_{U_{21}}(\cdot) = U_{21}(\cdot)U_{21}^{-1} \tag{82}$$

obeying

$$\gamma_1(U_{21}) = \varepsilon_\zeta \cdot U_{21} \tag{83}$$

where ε_ζ is a parameter depending only on the sector ζ and can only take the values ± 1 in $n \geq 2$ space dimensions.

Since the field algebra is an extension of the algebra of observables, an automorphism of the latter may be written as

$$\gamma_1(\cdot) = \psi_1(\cdot)\psi_1^{-1} \tag{84}$$

where ψ_1 is an unitary element of the former one. Consequently, we are allowed to set

$$U_{21} = \psi_2\psi_1^{-1} \tag{85}$$

Inserting equation (84) and equation (85) into equation (83) and multiplying both sides with $\psi_1\psi_1$ onto the right, we obtain the familiar spacelike commutation relations of field operators:

$$\psi_1\psi_2 = \varepsilon_\zeta\psi_2\psi_1 \tag{86}$$

Evidently this discussion suggests a natural choice for a quasi-configuration space: Let $\mathcal{O}_1, \ldots, \mathcal{O}_N$ be the localization regions (localized in a fixed time slice in Minkowski space for simplicity) of spacelike disjoint localized automorphisms $\gamma_1, \ldots, \gamma_N$ leading from the vacuum to the sector

$$\zeta = 1 \tag{87}$$

where

$$\varepsilon := \varepsilon_\zeta\big|_{\zeta=1} \tag{88}$$

denotes the associated DHR statistics parameter; then, with

$$\mathbf{x}_1 \in \mathcal{O}_1, \ldots, \mathbf{x}_N \in \mathcal{O}_N \tag{89}$$

we define

$$\omega_{\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}} := \omega_0 \circ \gamma_1 \cdots \gamma_N \tag{90}$$

and

$$\left|\Psi^{(\varepsilon)}_{(\mathbf{x}_1,\ldots,\mathbf{x}_N)}\right\rangle := \psi_1 \cdots \psi_N |\Psi_0\rangle \tag{91}$$

where ω_0 is the vacuum state and Ψ_0 a representing vector in a Hilbert space H constructed by making use of the GNS procedure (Haag, 1970). The family of ω's considered generates a quasi-configuration space $Q_{\mathrm{uasi}} \subset \mathbf{G}_1(H)$ homotopically equivalent to $C_N(\mathbf{R}^n)$. While because of equation (80) we have

$$\omega_{\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}} = \omega_{\{\mathbf{x}_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(N)}\}} \tag{92}$$

we get from equation (86)

$$\left|\Psi^{(\varepsilon)}_{(\mathbf{x}_1,\ldots,\mathbf{x}_N)}\right\rangle = \varepsilon^{\mathrm{sign}(\sigma)} \cdot \left|\Psi^{(\varepsilon)}_{(\mathbf{x}_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(N)})}\right\rangle \tag{93}$$

The equivalence classes of systems of local section on Q_{uasi}, i.e., the possible classes of equivariant state vectors

$$|\Psi^{\chi}_{(\mathbf{x}_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(N)})}\rangle = \chi(\sigma)\cdot|\Psi^{\chi}_{(\mathbf{x}_1,\ldots,\mathbf{x}_N)}\rangle \tag{94}$$

are in 1-to-1 correspondence to the elements of

$$\pi^1(Q_{\text{uasi}})\hat{} = \pi_1(C_N(\mathbf{R}^3))\hat{} = \mathbf{Z}_2 \tag{95}$$

and correspond for $n \geq 3$ to the admissible values of the DNR statistics parameter. This means that for every character χ we are able to find an ε such that with

$$\chi(\sigma) = \varepsilon^{\text{sign}(\sigma)} \tag{96}$$

we have the correspondence

$$|\Psi^{\chi}_{(\mathbf{x}_1,\ldots,\mathbf{x}_N)}\rangle = |\Psi^{(\varepsilon)}_{(\mathbf{x}_1,\ldots,\mathbf{x}_N)}\rangle \tag{97}$$

This is no longer true in two space dimensions. Because of

$$\pi_1(C_N(\mathbf{R}^2))\hat{} = \mathbf{U}(1) \tag{98}$$

we now have systems of local sections for which we cannot find an ε parameter, i.e., no sector with the corresponding anomalous statistics. Admitted are only state vectors for which the character in

$$|\Psi^{\chi}_{(\mathbf{x}_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(N)})[\text{braid}](\mathbf{x}_1,\ldots,\mathbf{x}_N))}\rangle = \chi(\sigma)\cdot|\Psi^{\chi}_{(\mathbf{x}_1,\ldots,\mathbf{x}_N)[\text{trivial braid}](\mathbf{x}_1,\ldots,\mathbf{x}_N))}\rangle \tag{99}$$

is chosen in such a way that

$$|\Psi^{(\varepsilon)}_{(\mathbf{x}_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(N)})}\rangle = \varepsilon^{\text{sign}(\sigma)}\cdot|\Psi^{(\varepsilon)}_{(\mathbf{x}_1,\ldots,\mathbf{x}_N)}\rangle \tag{100}$$

holds. The notation [braid] indicates that the points of the universal covering of $C_N(\mathbf{R}^2)$ are not simply N-tuples $(\mathbf{x}_1,\ldots,\mathbf{x}_N)$, $\mathbf{x}_i \in \mathcal{O}$, but in addition are characterized by a specification of a homotopy class of braids defined in reference to a fixed base point.

There is a certain ambiguity in the choice of a quasi-configuration space: Its points parametrize all physical operations which are of interest for the "comprehensive analysis" of the problem. This choice fixes all possible topological quantum numbers. If we do not restrict ourselves to the transformations of spacelike disjoint automorphisms leaving the localization supports disjoint and include all inner morphisms of a certain (homotopically trivial) region, then we will have a parametrization of the inner morphisms by themselves, so that we get no nontrivial characters. On the other hand, the choice of $C_N(\mathbf{R}^2)$ is too restrictive: We get a too rich "spectrum" of toplogogical quantum numbers, for which we cannot find an acceptable physical interpretation.

The analysis of the preceding section suggests we choose a quasi-configuration space of positive and negative objects, namely with

$$\mathbf{x}_1 \in \mathcal{O}_1, \ldots, \mathbf{x}_{\mathcal{P}} \in \mathcal{O}_{\mathcal{P}}, \qquad \mathbf{y}_1 \in \mathcal{O}'_1, \ldots, \mathbf{y}_{\mathcal{N}} \in \mathcal{O}'_{\mathcal{N}}, \qquad N = \mathcal{N} - \mathcal{P} \tag{101}$$

where $\mathbf{x}_1 \in \mathcal{O}_1, \ldots, \mathbf{x}_{\mathcal{P}} \in \mathcal{O}_{\mathcal{P}}$ and $\mathbf{y}_1 \in \mathcal{O}'_1, \ldots, \mathbf{y}_{\mathcal{N}} \in \mathcal{O}'_{\mathcal{N}}$ are mutually disjoint, we set

$$\omega_{(\{\mathbf{x}_1,\ldots,\mathbf{x}_{\mathcal{P}}\},\{\mathbf{y}_1,\ldots,\mathbf{y}_{\mathcal{N}}\})} = \omega_0 \circ \gamma_1 \cdots \gamma_{\mathcal{P}} \bar{\gamma}_1 \cdots \bar{\gamma}_{\mathcal{N}} \tag{102}$$

where $\bar{\gamma}_1, \ldots, \bar{\gamma}_{\mathcal{N}}$ leads to the conjugated sector, and

$$|\Psi_{[((\mathbf{x}_1,\ldots,\mathbf{x}_{\mathcal{P}}),(\mathbf{y}_1,\ldots,\mathbf{y}_{\mathcal{N}}))]}\rangle = \psi_1 \cdots \psi_{\mathcal{P}} \bar{\psi}_1 \cdots \bar{\psi}_{\mathcal{N}} |\Psi_0\rangle \tag{103}$$

where the automorphisms are chosen in such a way, that

$$\gamma_i \bar{\gamma}_i = \mathbf{1} \tag{104}$$

If states and state vectors are defined in an analogous way, we will get, because of

$$\pi_1(C^{\pm}_N(\mathbf{R}^2))^{\wedge} = \mathbf{Z}_2 \tag{105}$$

a 1-to-1 correspondence between the characters of the fundamental group of the quasi-configuration space and the admitted values of the ε parameter.

The latter argument is of a purely heuristic nature. The central problem remaining to be solved is to find a convenient topologization of the "pair annihilation homotopy." In the remaining part of the paper I show that herein lies the key to the topological spin–statistics relation.[17]

Before coming to this point, I define a spin parameter η analogous to the topologically interpreted DHR statistics parameter ε. Let α_Λ be an automorphism of the observable algebra representing a spacelike rotation Λ (viewed as a Lorentz transformation); then there is an inner automorphism σ_{U_Λ} obeying

$$\alpha_\Lambda \gamma \alpha_\Lambda^{-1} = \sigma_{U_\Lambda} \gamma =: \gamma_\Lambda \tag{106}$$

with

$$\sigma_{U_\Lambda}(\cdot) = U_\Lambda(\cdot) U_\Lambda \tag{107}$$

The set $\{\omega_0 \circ \gamma_\Lambda\}$ may be interpreted as a quasi-configuration space homeomorphic to the configuration space $\mathbf{SO}(3)$ of the Bopp–Haag spin model. The role of the local wave function system is now played by a system of local unitary-valued sections fulfilling "constant phase" compatibility conditions. The right choice of the cohomology class is determined by the correct transformation law of the corresponding field operator, symbolically

[17] The discussion of the case describing charges of the second kind [cf. the analysis in Buchholz and Fredenhagen, 1982)] will be presented elsewhere.

written as

$$U_{2\pi}\psi = \eta\psi \tag{108}$$

where η must be $+1$ (integer spin) or -1 (half-odd spin).

Thus we are able to construct and lift quasi-configuration spaces of rigidly moving automorphisms labeled by the elements of $C^{\pm}_{N\mathbf{SO}(3)}(\mathbf{R}^3)$. An element of such a space is written as

$$q = \gamma_{(\mathbf{x}_1,\mathbf{e}_1)} \cdots \gamma_{(\mathbf{x}_{\mathcal{N}},\mathbf{e}_{\mathcal{N}})}\bar{\gamma}_{(\mathbf{y}_1,\mathbf{e}'_1)} \cdots \bar{\gamma}_{(\mathbf{y}_{\mathcal{P}},\mathbf{e}'_{\mathcal{P}})} \tag{109}$$

3.3. Dirac's String Game Visualizes Spin, Statistics, and the Connection between Them

In his famous lectures Dirac used to demonstrate the doubly connectedness of the 3-space rotation group $\mathbf{SO}(3)$ by a simple model: Two pairs of scissors are connected to each other by at least three rubber bands. Rotating one pair through 2π, the threads become tangled, such that there is no way to disentangle them. Rotating the same pair through a further 2π, the rubber bands become even more tangled, but now it is possible to disentangle them, so that we recover the original situation (cf. Figure 4).

There are many variations of the theme, i.e., of Dirac's string game; a very popular treatment due to McDonald can be found in Misner *et al.*, 1973). The reader is also referred to Newman (1942), Bolker (1973), Rieflin (1979), Guerra and Marra, (1983, 1984), and Mickelsson, (1984). That Dirac strings are related to the double connectedness of $\mathbf{SO}(3)$ is by no means trivial and was only clarified by Fadell (1962) by using the theory of braids. In topological terms Dirac's string game visualizes the canonical 1-to-1 homomorphism of $\pi_1(\mathbf{SO}(3))$ into $\pi_1(D_N(\mathbf{S}^2))$, $N \geq 3$.

It is often overlooked that this fascinating model not only provides a topological representation of spin, but also visualizes statistics and the connection between the former and the latter. In fact, a 2π rotation may

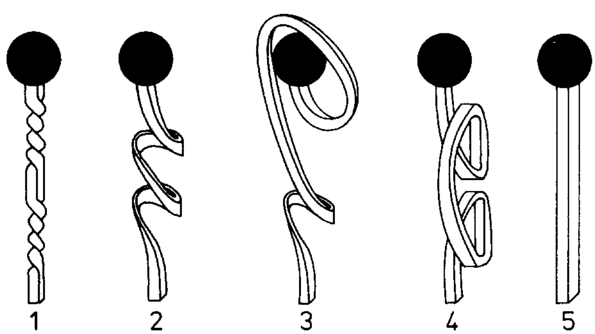

Fig. 4. An easy-to-visualize version of Dirac's string model for spin.

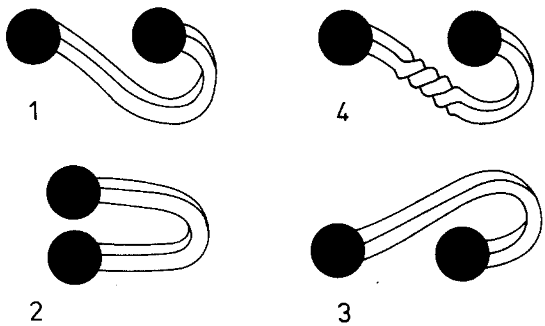

Fig. 5. Dirac's string model for the connection of spin and statistics.

be compensated by an orientation-preserving exchange of the two pairs of scissors. By the way, the pairs of scissors may be replaced by balls and the $N \geq 3$ rubber bands by one fat rubber band (e.g., with a squarelike cross section connecting the balls). Figure 5 shows this simple mechanism that relates 2π rotation and exchange, i.e., spin and statistics.

One question now arises: What physical objects are represented by these strings or this one fat rubber band?

3.4. The Finkelstein–Rubinstein Mechanism

In kink models invented by Finkelstein and Misner one has indeed a "realization" of Dirac's string game. In this framework the rubber bands do not play the role of fat, nonobservable strings in the space of perception (through structures of this type may have some relevance in quantum field theory), but they represent the trajectories of certain extended objects, namely, the kinks.

Here I briefly review the fundamental notions of kink theory (Finkelstein, 1966, 1969; 1968). In general one considers configuration spaces of single-valued continuous classical field configurations taking values in a finite-dimensional *field manifold* Φ and defined, e.g., on space of perception $\mathbf{R}^3$ subject to certain boundary conditions. This function space is topologized by means of the *compact open topology.*[18]

In *kink theories of the first kind* such a configuration space is written as

$$\begin{aligned} &\mathscr{C}((\mathbf{R}^3, \infty), (\Phi, \phi_0)) \\ &\quad := \{\varphi\colon \mathbf{R}^3 \to \Phi \mid \varphi \in \mathscr{C}(\mathbf{R}^3, \Phi);\ \lim_{|\mathbf{x}| \to \infty} \varphi(\mathbf{x}) = \phi_0;\ \phi_0 \in \Phi\} \end{aligned} \tag{111}$$

[18]For every compact subset $K \subset \mathbf{R}^3$ and every open subset $U \subset \Phi$ the sets

$$W(K, U) = \{f \in \mathscr{C}(\mathbf{R}^3, \Phi)\colon f(K) \subset U\} \tag{110}$$

form a basis of a topology.

By using stereographic projection, we have the homeomorphisms

$$\mathscr{C}((\mathbf{R}^3, \infty), (\Phi, \phi_0)) \cong \mathscr{C}((\mathbf{S}^3, \mathcal{N}), (\Phi, \phi_0)) \cong \mathscr{C}((\mathbf{I}^3, \partial\mathbf{I}^3), (\Phi, \phi_0)) \quad (112)$$

where $\mathcal{N}$ is the north pole of the 3-sphere $\mathbf{S}^3$, $\mathbf{I}^3$ the 3-cube, and the $\partial\mathbf{I}^3$ its boundary surface. Configuration spaces of this kind are distinguished by the fact that they fall into connected components standing in 1-to-1 correspondence to the homotopy classes of fields

$$\pi_0(\mathscr{C}((\mathbf{R}^3, \infty), (\Phi, \phi_0))) = \pi_3(\Phi, \phi_0) = \pi_3(\Phi) \quad (113)$$

There is a theorem by Whitehead (1953) stating that all connected components $\mathscr{C}^{(g)}(\cdots)$ of $\mathscr{C}(\cdots)$ have the same homotopy type. Thus, we have

$$\pi_1(\mathscr{C}^{(g)}((\mathbf{I}^3, \partial\mathbf{I}^3), (\Phi, \phi_0))) = \pi_1(\mathscr{C}^{(0)}((\mathbf{I}^3, \partial\mathbf{I}^3), (\Phi, \phi_0))) \quad (114)$$

$$= \pi_1(\mathscr{C}^{(0)}((\mathbf{I}^3, \partial\mathbf{I}^3), (\Phi, \phi_0)), \varphi(\cdot) \equiv \phi_0) \quad (115)$$

$$= \pi_4(\Phi, \phi_0) \quad (116)$$

$$= \pi_4(\Phi) \quad (117)$$

In case of a nontrivial $\pi_4(\Phi)$ we expect multiple-valued quantizations.

In case of a *kink model of the second kind* we deal with configuration spaces

$$\mathscr{C}((\mathbf{R}^3, \mathbf{S}^2_\infty), (\Phi, \Phi^\partial))$$

$$= \left\{ \varphi\colon \mathbf{R}^3 \to \Phi \,\middle|\, \varphi \in \mathscr{C}(\mathbf{R}^3, \Phi);\ \lim_{|\mathbf{x}| \to \infty} \varphi(\mathbf{x}) = \hat{\varphi}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right);\ \hat{\varphi}\colon \mathbf{S}^2_\infty \to \Phi^\partial \right\} \quad (118)$$

Taking the new type of boundary condition into account, we obtain the following homeomorphisms:

$$\mathscr{C}((\mathbf{R}^3, \mathbf{S}^2_\infty), (\Phi, \Phi^\partial)) \cong \mathscr{C}((\mathbf{D}^3, \partial\mathbf{D}^3), (\Phi, \Phi^\partial)) \cong \mathscr{C}((\mathbf{I}^3, \partial\mathbf{I}^3), (\Phi, \Phi^\partial)) \quad (119)$$

where $\mathbf{D}^3$ is the 3-ball with boundary $\partial\mathbf{D}^3$. In addition, we have

$$\pi_0(\mathscr{C}((\mathbf{R}^3, \mathbf{S}^2_\infty), (\Phi, \Phi^\partial))) = \pi_3(\Phi, \Phi^\partial) \quad (120)$$

In case of a contractible Φ we get

$$\pi_3(\Phi, \Phi^\partial) = \pi_0(\mathscr{C}(\mathbf{S}^2, \Phi^\partial)) = \pi_2(\Phi^\partial) \quad (121)$$

Kinks of the first (resp. second) kind may be interpreted as classical realizations of charges of the first (resp. second) kind. An example for the first type is the Skyrme soliton (Skyrme, 1955, 1958, 1959, 1961*a*,*b*, 1962; Perring and Skyrme, 1962), and for the second type the 'tHooft–Polyakov monopole ('tHooft, 1974; Polyakov, 1974).[19]

[19]The computation of the fundamental group of the configuration space is more difficult in the case of kinks of the second kind, since here we have no theorem à la Whitehead. Ringwood and Woodward (1981, 1982) did some computations for the 'tHooft–Polyakov monopole.

Field configurations are called *null-homotopic* if and only if they lie in the homotopy class of the constant field

$$\varphi_{\text{vacuum}}(\mathbf{x}) \equiv \phi_0 \tag{122}$$

also known as the *vacuum configuration*. They are so-called localizing homotopies, which allow one to deform a kink field configuration in such a way that it has its support[20] in a ball-like region in the case of the first kind and in a conelike region in the case of the second kind. In this context we talk about *simply localized kinks*. Analogously, one can also define *multiply localized kinks*. A simply localized kink of charge one may be interpreted as an *extended object* that carries exactly one unit of charge.

Note that the definition of homotopy allows the continuous creation and annihilation of kink–antikink pairs. Details are found below. In the following let us restrict ourselves to the investigation of kinks of the first kind. Problems incorporating kinks of the second kind may be reduced to a "first-kind problem" by decrementing the number of dimensions of the space of perception.

3.4.1. Kink Trajectories as Fat Strings

A simply localized kink may be rigidly translated and rotated. In addition, "multikinks" may be exchanged, and that is exactly the way to study the spin and statistics of these objects.

A trajectory

$$t \mapsto \mathbf{x}(t) \tag{123}$$

in the space of perception $\mathbf{R}^3$ may be viewed as a certain string

$$t \mapsto (t, \mathbf{x}(t)) \tag{124}$$

in space-time[21] of perception

$$\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3 \tag{125}$$

This string may be homotopically deformed to any line parametrized as

$$\tau \mapsto (t(\tau), x(\tau)) \tag{126}$$

For instance, it is possible for the line to go backward in time.

To a field configuration describing a multiple-valued quantizable localized kink in space of perception $\boldsymbol{R}^3$ we naturally attach a 3-bein

$$\mathbf{e}^{(3)} = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) \tag{127}$$

[20] By support of a kink we mean this region in the space of perception which essentially looks different from the vacuum configuration.

[21] Indeed, the homotopy parameter controlling rotation and exchange may be viewed as a time parameter parametrizing an adiabatic process.

representing its orientation. It is useful to think of a 3-bein attached to the localization center of the kink. If we only consider rigid motions, we will be able to discuss all features on the 3-bein level.

Next consider a kink trajectory interpreted as a fat string in space-time of perception $\mathbf{R}^4$. With the help of a homotopic deformation it is possible to deform the trajectory in such a way that the kink center trajectory is smooth and the intersections of the normal hyperplanes of the latter with the full kink trajectory are copies of the field configuration describing "a kink in rest," but rotated in $\mathbf{R}^4$. The orientation of such a copy is conveniently described by a 4-bein

$$\mathbf{e}^{(4)} = (\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) \tag{128}$$

where $\mathbf{n}_0$ is tangentially aligned to the kink center trajectory, while $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are living in the normal hyperplane.

Expressed in other words, the essential structure is given by a trajectory $t \mapsto (t, \mathbf{x}(t))$ and an associated family

$$\mathbf{e}^{(4)}(t) = (\mathbf{n}_0(t), \mathbf{n}_1(t), \mathbf{n}_2(t), \mathbf{n}_3(t)) \tag{129}$$

of tangentially aligned 4-beins obeying

$$\mathbf{n}_0(t) \Big\| \frac{d}{dt}(t, \mathbf{x}(t)) \tag{130}$$

A smoothed kink–antikink pair creation or a smoothed kink–antikink pair annihilation process may be viewed as a kink trajectory reversing its time direction. On the level of moving 4-beins it looks as follows: We consider any strings in space-time $(t(\tau), \mathbf{x}(\tau))$, to which 4-beins $\mathbf{e}^{(4)}(t)$ are attached according to

$$\mathbf{n}_0(\tau) \Big\| \frac{d}{d\tau}(t(\tau), \mathbf{x}(\tau)) \tag{131}$$

A π rotation in the space-time of perception reversing the direction of the "time basis vector" $\mathbf{n}_0$ of the 4-bein $\mathbf{e}^{(4)}$ acts on the 3-bein spanned by the "space basis vectors" $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ like a reflection: e.g., it leaves $\mathbf{n}_2$, $\mathbf{n}_3$ invariant and changes the sign of $\mathbf{n}_1$.[22] From a homotopic standpoint this 3-bein reflection is nothing but a homotopic conjugation, namely a switch to a field configuration belonging to the inverse homotopy class. Foliating the space-time of perception by spaces of perceptions, we get snapshots of a process in which, e.g., for a kink–antikink annihilation it seems that the former and the latter eat each other. Hence, processes incorporating

[22]In the famous analytic proof of the spin–statistics theorem one also makes use of the fact that in the Euclidean 4-space rotation group (resp. in the complexified Lorentz group) there exists a continuous path from $\mathbf{1}$ to $\mathbf{PT}$. We also can say that a noncontractible 2π rotation loop $\mathbf{1} \to \pi \to 2\pi = \mathbf{1}$ can be continuously deformed into a noncontractible loop $\mathbf{1} \to \mathbf{PT} \to \mathbf{1}$.

rotations, exchanges, and pair creations that start and end with a vacuum configuration are described by an assembly of closed fat strings in space-time of perception $\mathbf{R}^4$. These fat strings can be replaced by linelike strings to which a family of tangentially aligned 4-beins is associated. The tangential alignment of the 4-beins along the closed string in $\mathbf{R}^4$ defines a loop in $\mathbf{SO}(4)$. Accordingly, there are only two classes of twists for fat strings in four dimensions. Alternatively, we are allowed to consider one or more open fat string with fixed ends. Also in this case we have only two homotopy classes of paths. The tangentially aligned moving 4-beins canonically define a four-string braid, giving a generalization of Dirac's string game to four dimensions incorporating at least four strings.

3.4.2. The Homotopy Extension Proof

The homotopy spin–statistics theorem as formulated by Finkelstein and Rubenstein (1968) reads as follows.

Theorem 3.1. In kink models odd-half spin is possible if and only if Fermi–Dirac statistics is possible.

There is a homotopy $\varphi_{\text{rot to triv}}(\mathbf{x}, t, u)$ deforming a loop $\varphi_{\text{rot}}(\mathbf{x}, t)$, in which one kink is rotated through 2π, into a trivial loop $\varphi_{1\,\text{triv}}(\mathbf{x}, t)$, in which the kink is left in rest, *if and only if* there is a homotopy $\varphi_{\text{exch to triv}}(\mathbf{x}, s, t)$ deforming a loop $\varphi_{\text{exch}}(\mathbf{x}, t, u)$, in which two identical kinks are exchanged, into a trivial loop $\varphi_{2\,\text{triv}}(\mathbf{x}, t)$, in which the two kinks are left at rest. The processes considered may be localized in a 3-cube $\mathbf{I}^3$ and hence the homotopies in question, if they exist, define continuous functions on the 5-cube $\mathbf{I}^3 \times [0, 1] \times [0, 1]$ whose boundary values on $\partial\mathbf{I}^5$ are given by

$$\begin{aligned}
\varphi_{\text{rot to triv}}(\mathbf{x}, s, 0) &= \varphi_{\text{rot}}(\mathbf{x}, s)\\
\varphi_{\text{rot to triv}}(\mathbf{x}, s, 1) &= \varphi_{1\,\text{triv}}(\mathbf{x}, s)\\
\varphi_{\text{rot to triv}}(\mathbf{x}, 0, t) &= \varphi_{1\,\text{triv}}(\mathbf{x}, t)\\
\varphi_{\text{rot to triv}}(\mathbf{x}, 1, t) &= \varphi_{1\,\text{triv}}(\mathbf{x}, t)\\
\varphi_{\text{rot to triv}}(\partial\mathbf{I}_0^3, s, t) &= \phi_0\\
\varphi_{\text{exch to triv}}(\mathbf{x}, s, 0) &= \varphi_{\text{exch}}(\mathbf{x}, s)\\
\varphi_{\text{exch to triv}}(\mathbf{x}, s, 1) &= \varphi_{2\,\text{triv}}(\mathbf{x}, s)\\
\varphi_{\text{exch to triv}}(\mathbf{x}, 0, t) &= \varphi_{2\,\text{triv}}(\mathbf{x}, t)\\
\varphi_{\text{exch to triv}}(\mathbf{x}, 1, t) &= \varphi_{2\,\text{triv}}(\mathbf{x}, t)\\
\varphi_{\text{exch to triv}}(\partial\mathbf{I}_0^3, s, t) &= \phi_0
\end{aligned} \tag{132}$$

The homotopy extension theorem states that if the homotopic restrictions $\varphi_{\mathrm{rot\,to\,triv}}|_{\partial\mathbf{I}^5}$ and $\varphi_{\mathrm{exch\,to\,triv}}|_{\partial\mathbf{I}^5}$ are given, the existence of an extension of $\varphi_{\mathrm{rot\,to\,triv}}|_{\partial\mathbf{I}^5}$ onto $\mathbf{I}^5$ is equivalent to the existence of an extension of $\varphi_{\mathrm{exch\,to\,triv}}|_{\partial\mathbf{I}^5}$ onto $\mathbf{I}^5$. The homotopy of both functions defined on the boundary is verified by rubber band techniques and easily visualized: Neglecting one dimension of the space of perception, we draw the boundary of the 4-cube $\mathbf{I}^2\times[0,1]\times[0,1]$ with the corresponding fat strings inserted (cf. Figure 6).

3.4.3. A Different View: Deforming a Rotation into an Exchange by Use of Pair Creation

Alternatively, one can interpret the closed, fat strings of Figure 6 as trajectories of kinks: The one string describes a process in which a kink-antikink pair is created and, after a 2π-rotation of the kink, is annihilated; the other describes a partner exchange process. It is easily seen that they are homotopic. A simplified version of the partner exchange loop is the *twisted pair loop*, a simple kink-antikink creation-annihilation loop, in which the relative orientations of the partners during the creation and annihilation are changed (Figure 7).

Since kink and antikink trajectories can mutually extinguish each other, one is able to get an elegant visualization of a deformation of a one-kink rotation into a two-kink exchange (Figure 8).

By using the same techniques, one immediately sees that the complete problem may be reduced to the homotopy between the twisted pair loop and a rotation loop (Figure 7). It is possible to find a natural analog of this simple homotopy in a general quantum field-theoretic context?

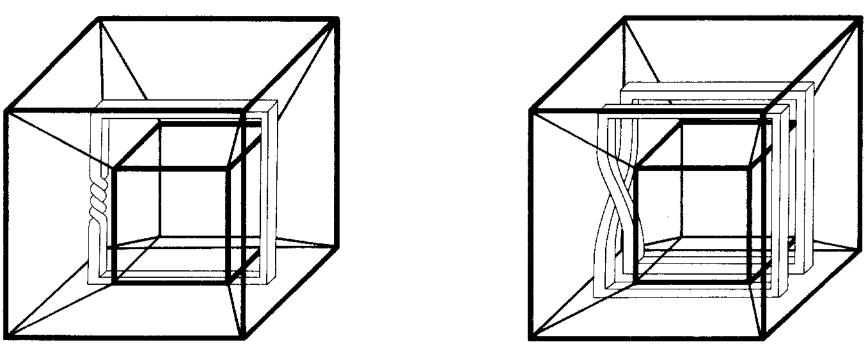

Fig. 6. $\varphi_{\mathrm{rot\,to\,triv}}|_{\partial\mathbf{I}^5}$ versus $\varphi_{\mathrm{exch\,to\,triv}}|_{\partial\mathbf{I}^5}$.

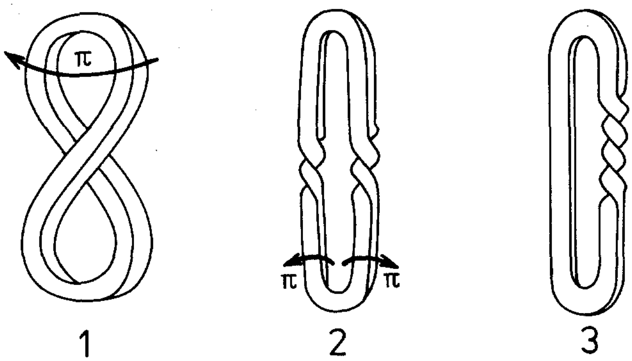

Fig. 7. $\varphi_{\text{twisted pair loop}}$ homotopic to $\varphi_{\text{rotation loop}}$.

3.5. Skyrmions: The Replacement of Homotopic by Algebraic Pair Creations

The close analogy between the mechanistic configuration spaces of rigid, noncoinciding objects and the configuration space à la Doplicher-Haag-Roberts is due to the commutativity of disjoint spacelike supported automorphisms. It becomes rather tricky when we wish to incorporate pair creation and annihilation processes that are necessary to deform rotation loops into exchange loops. Unfortunately, the replacement of the particle

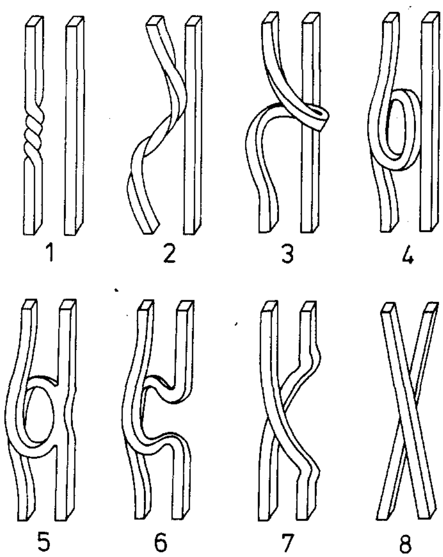

Fig. 8. $\varphi_{\text{rot to exch}}$.

configuration models by kink field configuration models does not suffice, since the pair annihilation (resp. creation) by the homotopic process of "mutual eating" (resp. "mutual uneating") has no counterpart in algebraic quantum field theory. However, in certain kink models, where the classical fields take their values in a group manifold, field configurations may be multiplied pointwise and we can show that the purely algebraic property of the noncommutativity of this multiplication eventually gives rise to nontriviality of certain loops. This justifies the following conjecture, not yet proven.

Conjecture 3.1. There exists a quasi-configuration space containing all products of rigidly moving spacelike disjoint automorphisms of class one and their conjugate

$$\gamma_{(\mathbf{x}_1,\mathbf{e}_1)} \cdots \gamma_{(\mathbf{x}_{\mathscr{P}},\mathbf{e}_{\mathscr{P}})} \bar{\gamma}_{(\mathbf{y}_1,\mathbf{e}'_1)} \cdots \bar{\gamma}_{(\mathbf{y}_{\mathscr{P}},\mathbf{e}'_{\mathscr{P}})} \tag{133}$$

and all overlapping configurations of the type

$$\cdots \gamma_{(\mathbf{x}_i,\mathbf{e}_i)} \bar{\gamma}_{(\mathbf{y}_j,\mathbf{e}'_j)} \cdots \tag{134}$$

and

$$\cdots \bar{\gamma}_{(\mathbf{y}_j,\mathbf{e}'_j)} \gamma_{(\mathbf{x}_i,\mathbf{e}_i)} \cdots \tag{135}$$

having $\mathbf{Z}_2$ as its fundamental group. In particular, the homotopically defined statistics parameter coincides with the homotopically defined spin parameter.

Incorporating the coinciding configurations into configuration models of pointlike objects destroys their manifold structure. This can be cured by counting diagonal configurations doubly and distinguish them clearly from each other.

The $\mathbf{SU}(2)$-Skyrme model provides a nice realization of this principle, which here can be tested easily, since algebraically *and* homotopically defined processes are possible *at the same time.*

Let φ_{kink} denote a localized one-kink field configuration obeying

$$\varphi_{\text{kink}}(\mathbf{x} \notin \mathbf{I}) = \mathbf{1} \tag{136}$$

where $\mathbf{I}$ is the space-centered 3-cube given by

$$\mathbf{I} = \left[-\frac{\varepsilon}{2}, +\frac{\varepsilon}{2}\right] \times \left[-\frac{\varepsilon}{2}, +\frac{\varepsilon}{2}\right] \times \left[-\frac{\varepsilon}{2}, +\frac{\varepsilon}{2}\right] \tag{137}$$

There are two ways to transform this kink configuration into an antikink configuration. One way is given by the *homotopic conjugation,*

$$\varphi_{\text{homotopic antikink}}(\mathbf{x}) = \varphi_{\text{kink}}(x_1, x_2, -x_3) \tag{138}$$

$$= \varphi_{\text{kink}}[\exp(i\pi\mathbf{n}_3 \times)(-\mathbf{x})] \tag{139}$$

$$:= \exp(i\pi\mathbf{L}_3)\mathbf{P}\varphi_{\text{kink}}(\mathbf{x}) \tag{140}$$

the other is defined by the *algebraic conjugation*,[23]

$$\varphi_{\text{algebraic antikink}}(\mathbf{x}) = [\varphi_{\text{kink}}(\mathbf{x})]^{-1} \tag{141}$$

$$=: \mathbf{C}\varphi_{\text{kink}}(\mathbf{x}) \tag{142}$$

Consider now the simplest noncontractible loop in the vacuum sector, the *twisted pair loop.* It is given by $(t \in [0, 3])$

$$\varphi_{\text{twisted pair loop}}(\mathbf{x}, t) := \begin{cases} \varphi_{\text{homotopic } LR \text{ creation}}(\mathbf{x}, t), & t \in [0, 1] \\ \varphi_{\text{antikink kink exchange}}(\mathbf{x}, t-1), & t \in [1, 2] \\ \varphi_{\text{homotopic } RL \text{ annihilation}}(\mathbf{x}, t-2), & t \in [2, 3] \end{cases} \tag{143}$$

whereby $(t \in [0, 1])$

$$\varphi_{\text{homotopic } LR \text{ creation}}(\mathbf{x}, t) := \begin{cases} \varphi_{\text{homotopic antikink}}(\mathbf{x} + (t - \frac{1}{2})2^{1/2}\varepsilon\mathbf{n}_3), & x_3 \leq 0 \\ \varphi_{\text{kink}}(\mathbf{x} - (t - \frac{1}{2})2^{1/2}\varepsilon\mathbf{n}_3), & x_3 \geq 0 \end{cases} \tag{144}$$

resp.

$$\varphi_{\text{homotopic } RL \text{ creation}}(\mathbf{x}, t) := \begin{cases} \varphi_{\text{kink}}(\mathbf{x} + (t - \frac{1}{2})2^{1/2}\varepsilon\mathbf{n}_3), & x_3 \leq 0 \\ \varphi_{\text{homotopic antikink}}(\mathbf{x} - (t - \frac{1}{2})2^{1/2}\varepsilon\mathbf{n}_3), & x_3 \geq 0 \end{cases} \tag{145}$$

and the corresponding annihilation processes are obtained from the creation processes by replacing t through $1-t$. The homotopy $\varphi_{\text{antikink kink exchange}}$ describes a process where the mutually disjoint localized antikink and kink are exchanged by performing rigid translations.

We now are able to prove the nontriviality of the algebraic analogue to the homotopically defined twisted pair loop $(r \in [0, 2])$:

$$\varphi_{\text{algebraically twisted pair loop}}(\mathbf{x}, t) := \begin{cases} \varphi_{\text{algebraic } LR \text{ creation}}(\mathbf{x}, t), & t \in [0, 1] \\ \varphi_{\text{algebraic } RL \text{ annihilation}}(\mathbf{x}, t-1), & t \in [1, 2] \end{cases} \tag{146}$$

whereby $(t \in [0, 1])$

$$\varphi_{\text{algebraic } LR \text{ creation}}(\mathbf{x}, t) \tag{147}$$

$$:= \varphi_{\text{algebraic antikink}}\left(\mathbf{x} + \frac{t}{2}2^{1/2}\varepsilon\mathbf{n}_3\right) \cdot \varphi_{\text{kink}}\left(\mathbf{x} - \frac{t}{2}2^{1/2}\varepsilon\mathbf{n}_3\right) \tag{148}$$

[23] We have seen that a continuous kink–antikink pair creation can be interpreted as a process in which one kink, moving in space-time, continuously reverses its time direction (and hence at least one space direction). This process may be viewed as a continuous path leading from the identity to $\exp(i\pi\mathbf{L}_3)\mathbf{T}$ or more correctly to $\exp(i\pi\mathbf{L}_3)\mathbf{PCT}$. It is no accident that the latter is to be identified with the well-known *modular conjugation* $\mathbf{J}$ (Bisognano and Wichmann, 1976).

resp.

$$\varphi_{\text{algebraic } RL \text{creation}}(\mathbf{x}, t) \tag{149}$$

$$:= \varphi_{\text{kink}}\left(\mathbf{x} - \frac{t}{2} 2^{1/2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{algebraic antikink}}\left(\mathbf{x} + \frac{t}{2} 2^{1/2} \varepsilon \mathbf{n}_3\right) \tag{150}$$

and again the corresponding annihilation processes are obtained by the replacement $t \mapsto 1 - t$. Since

$$\varphi_{\text{algebraic antikink}}\left(\mathbf{x} + \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{kink}}\left(\mathbf{x} - \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \tag{151}$$

$$= \varphi_{\text{kink}}\left(\mathbf{x} - \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{algebraic antikink}}\left(\mathbf{x} + \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \tag{152}$$

it is easy to see that the algebraic twisted pair loop may be homotopically deformed to a six-stage loop consisting of three stages (parametrized by $[0, 3]$),

$$\begin{array}{rl}
\mathbf{1}, & t = 0 \\
\varphi_{\text{algebraic } LR \text{ creation}}(\mathbf{x}, t), & t \in [0, 1] \\
\varphi_{\text{algebraic antikink}}\left(\mathbf{x} + \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{kink}}\left(\mathbf{x} - \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right), & t = 1 \\
\varphi_{\text{algebraic antikink}}\left(\exp[i(t-1)\pi \mathbf{n}_3 \times]\mathbf{x} + \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{kink}}\left(\mathbf{x} - \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right), & t \in [1, 2] \\
\varphi_{\text{homotopic antikink}}\left(\mathbf{x} + \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{kink}}\left(\mathbf{x} - \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right), & t = 2 \\
\varphi_{\text{homotopic } LR \text{ annihilation}}(\mathbf{x}, t - 2), & t \in [2, 3] \\
\mathbf{1}, & t = 3
\end{array} \tag{153}$$

and further three stages (for reasons of simplicity parametrized not by $t \in [3, 6]$, but $t \in [3, 0]$)

$$\begin{array}{rl}
\mathbf{1}, & t = 3 \\
\varphi_{\text{homotopic } LR \text{ annihilation}}(\mathbf{x}, t - 2), & t \in [3, 2] \\
\varphi_{\text{homotopic antikink}}\left(\mathbf{x} + \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{kink}}\left(\mathbf{x} - \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right), & t = 2 \\
\varphi_{\text{algebraic antikink}}\left(\exp[i(t-1)\pi \mathbf{n}_3 \times]\mathbf{x} + \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{kink}}\left(\mathbf{x} - \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right), & t \in [2, 1] \\
\varphi_{\text{algebraic antikink}}\left(\mathbf{x} + \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right) \cdot \varphi_{\text{kink}}\left(\mathbf{x} - \frac{\sqrt{2}}{2} \varepsilon \mathbf{n}_3\right), & t = 1 \\
\varphi_{\text{algebraic } RL \text{ creation}}(\mathbf{x}, t), & t \in [1, 0] \\
\mathbf{1}, & t = 0
\end{array} \tag{154}$$

Note that the fourth and fifth stages are exactly the inverse of the third and second stages, respectively. If the second up to the fifth stages, during which antikink and kink commute, are omitted, we just get the original algebraically twisted pair loop.

The homotopy class of the latter three-stage loop is not changed if we apply a global **PT** operation, which, after superponing a complementary π rotation of the antikink and kink during the second stage, yields a loop described by

$$\begin{aligned}
&\mathbf{1}, && t=0\\
&\varphi_{\text{algebraic } LR \text{ creation}}(\mathbf{x}, t), && t\in[0,1]\\
&\varphi_{\text{algebraic antikink}}\left(\mathbf{x}+\frac{\sqrt{2}}{2}\varepsilon\mathbf{n}_3\right)\cdot\varphi_{\text{kink}}\left(\mathbf{x}-\frac{\sqrt{2}}{2}\varepsilon\mathbf{n}_3\right), && t=1\\
&\varphi_{\text{algebraic antikink}}\left(\exp[i(t-1)\pi\mathbf{n}_3\times]\mathbf{x}+\frac{\sqrt{2}}{2}\varepsilon\mathbf{n}_3\right)\cdot\varphi_{\text{kink}}\left(\mathbf{x}-\frac{\sqrt{2}}{2}\varepsilon\mathbf{n}_3\right), && t\in[1,2]\\
&\varphi_{\text{homotopic antikink}}\left(\mathbf{x}+\frac{\sqrt{2}}{2}\varepsilon\mathbf{n}_3\right)\cdot\varphi_{\text{kink}}\left(\mathbf{x}-\frac{\sqrt{2}}{2}\varepsilon\mathbf{n}_3\right), && t=2\\
&\varphi_{\text{homotopic } RL \text{ annihilation}}(\mathbf{x}, t-2), && t\in[2,3]\\
&\mathbf{1}, && t=3
\end{aligned} \tag{155}$$

which differs from the first loop in that the homotopic LR annihilation is replaced by a homotopic RL annihilation. From the nontriviality of the twisted pair loop we conclude that each three-stage loop composing the six-stage loop considered lies in a different homotopy class and hence the complete loop is not trivial. Thus we close with a skyrmion model version of Conjecture 3.1.

Lemma 3.1. There exists a configuration space containing all products of rigidly moving disjoint, singly charged skyrmions and antiskyrmions and all doubly counted, overlapping configurations having $\mathbf{Z}_2$ as its fundamental group. In particular, exchange loops and 2π rotation loops are homotopic.

4. DISCUSSION

While our argument is on a heuristical level, the idea of introducing configuration spaces in algebraic field theory may lead to new insights into the relations between space-time operations and inner symmetries.

On one hand, the topology of a infinite-dimensional Hilbert space is too simple in the sense of the Kuiper theorem (Kuiper, 1965; Araki *et al.*, 1971) to provide us with homotopic invariants which can be related to superselection quantum numbers. On the other hand, the restriction of the quantum configuration space of rigidly moving automorphisms is not very natural, so we have to search for something in between, which may be called a *manifold of physically admitted* or *physically good states* or *morphisms.*

There are two complementary approaches, namely the "refinement" of the Hilbert space or operator algebra description toward finite-dimensional spaces as formulated in the *compactness criterion* of Haag and Swieca (1965) and extended to the *nuclearity idea* of Buchholz *et al.* (1986), which has much to do with a selection of localized states obeying a physical high-energy behavior), and, "alternatively," the "digitization of quantum mechanics" by Finkelstein and von Weizsäcker (cf. Finkelstein, 1969; Castell *et al.* 1975), namely the approximation of infinite-dimensional Hilbert spaces by huge tensor products of two-dimensional Hilbert spaces describing quantum mechanics over a binary alternative $\mathbf{u}_r$.[24] Intuitively, there is no doubt that homotopic invariants as well as Euclidean continuation argumentations are recovered if we do some (approximate) "finitization."

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[24]It is an interesting fact that there exists a spin-statistics theorem for $\mathbf{u}_r$'s.

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